1 The Runs Test for Randomness

Abstract

This article describes a nonparametric test procedure to determine if a series of data values is random in order or if there is a nonrandom pattern present. Examples from a time series, an SPC chart, and a linear regression model are presented.

Introduction

The runs test is a nonparametric test used to determine if the values in a time series or any other logical series are random or if there is a nonrandom pattern present. The test statistic is the number of runs of successes and failures observed in the data stream. If too few runs are observed there may be clustering of the successes and failures. If too many runs are observed there may be rapid cycling of the process.

Where the Technique is Used

The runs test can be used to check for:

- the presence of a pattern in a binary (success/failure) data series
- the presence of a pattern on a statistical process control (SPC) chart
- lack of fit of a regression model

Data

The data must be a single time series or some other logical sequential series. The data may be nominal, ordinal, or variable. A minimum of 9 points is required to apply the test. It must be possible to classify each data point into a binary class of successes or failures. Points that cannot be classified as successes or failures are ties and are excluded from the data set.

Assumptions

1. The data come from a single continuous stream.
2. A classification scheme can be designed to classify individual data values as successes or failures.

Hypotheses Tested

H_0 : The data stream is random
H_A : There is a pattern in the data stream

Procedure

1. Clearly identify the single data stream to be sampled.
2. Drawn sequential samples from the data stream. Record the data in the order they are taken.
3. Determine a classification scheme to group the data into binary classes of success or failure. Use the scheme to classify each observation taken. Be careful to preserve the order of the values. Points that cannot be classified as successes or failures are ties and should be removed from the data set.

4. Count the number of successes (n_1) and the number of failures (n_2). Then count the number of runs (r). A run is a succession of successes (failures) preceded and followed by a failure (success) or no observation.

5. If the sample is small ($n_1 + n_2 \leq 30$) consult the table of critical reject values in Figure 2 for the desired α level $\left(r_{\alpha/2} \text{ and } r_{1-\alpha/2}\right)$. If $r \leq r_{\alpha/2}$ or $r \geq r_{1-\alpha/2}$ then reject H_0, otherwise if $r_{\alpha/2} < r < r_{1-\alpha/2}$ then accept H_0.

   If the sample size is large ($n_1 \geq 10$ and $n_2 \geq 10$) then the sampling distribution of r is approximately normal with:

$$\mu_r = \frac{2n_1n_2}{n_1+n_2} + 1$$

   and

$$\sigma_r = \sqrt{\frac{2n_1n_2\left(2n_1n_2 - n_1 - n_2\right)}{\left(n_1+n_2\right)^2\left(n_1+n_2-1\right)}}$$

   The test statistic is:

$$z = \frac{\left(r \pm \frac{1}{2}\right) - \mu_r}{\sigma_r}$$

   where the $\pm\frac{1}{2}$ comes from the continuity correction. Use $+\frac{1}{2}$ if $r < \mu_r$ and use $-\frac{1}{2}$ if $r > \mu_r$. If $z < z_{\alpha/2}$ or $z > z_{1-\alpha/2}$ then reject H_0, otherwise if $z_{\alpha/2} \leq z \leq z_{1-\alpha/2}$ then accept H_0.

Example: *Determine if there is evidence that the following series of defective (d) and nondefective (n) parts is random or if there is a pattern in the data:*

$$nnnndnndnnnnndnnnndd$$

Use $\alpha = 0.05$.
Solution: *There are* $n_1 = 15$ *nondefective parts and* $n_2 = 5$ *defective parts. To make the runs easier to identify we can separate them with spaces and underline them:*

$$\underline{nnnn}\ \underline{d}\ \underline{nn}\ \underline{d}\ \underline{nnnnn}\ \underline{d}\ \underline{nnnn}\ \underline{dd}$$

The number of runs is $r = 8$. *From the table of critical reject values we find* $r_{0.025} = 4$ *and* $r_{0.975} = \infty$. *Since* $4 < 8 < \infty$ *we must accept* H_0 *and conclude that there is no evidence of a pattern in the data.*

Example: *It is suspected that there is a nonrandom pattern of points on a control chart. The following series of points above (a) and below (b) the center line of the chart is observed.*

$$aaaaabbaaaaatabbbaaaatbbaaaaaabaatabbbaaaaaab$$

Several t's correspond to points that fall exactly on the center line of the chart. Test for the presence of a nonrandom pattern using $\alpha = 0.05$.
Solution: *The points that fall exactly on the center line of the chart are ties and are removed from the data set. There are* $n_1 = 30$ *points above the center line and* $n_2 = 12$ *points below the center line. The number of runs is* $r = 12$. *The expected number of runs is:*

$$\mu_r = \frac{2 \times 30 \times 12}{30+12} + 1 = 18.1$$

and the corresponding standard deviation is:

$$\sigma_r = \sqrt{\frac{2\,(30)\,(12)\,(2 \times 30 \times 12 - 30 - 12)}{(30+12)^2\,(30+12-1)}} = 2.60$$

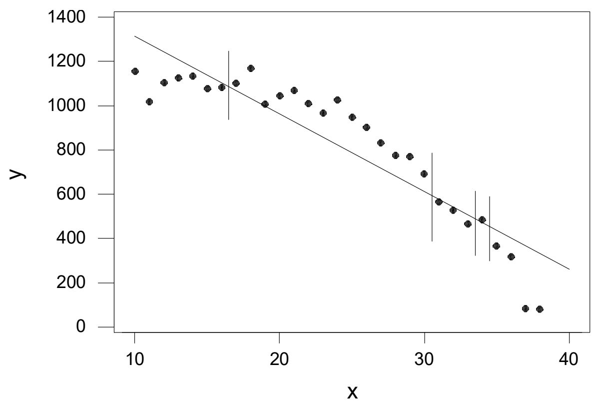

Figure 1: Scatter Plot, Linear Fit, and Runs About the Fit

Because $z_{0.025} = 1.96$ *the accept region for* H_0 *is given by:*

$$\Phi\left(-1.96 < z < 1.96\right) = 0.95$$

The test statistic is:

$$z = \frac{\left(12 + \frac{1}{2}\right) - 18.1}{2.60} = -2.15$$

which falls outside of the acceptance interval, so we must reject H_0 *and conclude that the data are not random.*

Example: *The linear fit of* y *vs.* x *in Figure 1 has a correlation coefficient of* $R^2 = 0.84$ *but it is suspected that there might be curvature in the data. Test to see if there is evidence of lack of fit.*
Solution: *We want to test the hypotheses* H_0 *: The data are randomly scattered about the regression line versus* H_A *: There is evidence of curvature or lack of fit. To test these hypotheses we will use the runs test. Each point is classified by its position above or below the fitted line. Several points fall near the regression line but none fall exactly on the line. If any points did fall right on the line they would be ties and would be removed from the data set. The number of points below the fitted line is* $n_1 = 14$ *and the number of points above the fitted line is* $n_2 = 15$*. The number of runs is* $r = 5$*. From the table of critical values we find that* $r_{0.025} = 9$*. Since* $(r = 5) \leq (r_{0.025} = 9)$ *we must reject* H_0 *and conclude that there is evidence of lack of fit.*

References

Freund and Simon, *Modern Elementary Statistics*, 9th Ed., Prentice-Hall, 1997.
Freund and Walpole, *Mathematical Statistics*, 3rd Ed., Prentice-Hall, 1980.

Critical Lower Reject Values of r for Runs Test ($\alpha/2=0.025$)

		n2													
		2	3	4	5	6	7	8	9	10	11	12	13	14	15
	2											2	2	2	2
	3					2	2	2	2	2	2	2	2	2	3
	4				2	2	2	3	3	3	3	3	3	3	3
	5			2	2	3	3	3	3	3	4	4	4	4	4
	6		2	2	3	3	3	3	4	4	4	4	5	5	5
	7		2	2	3	3	3	4	4	5	5	5	5	5	6
n1	8		2	3	3	3	4	4	5	5	5	6	6	6	6
	9		2	3	3	4	4	5	5	5	6	6	6	7	7
	10		2	3	3	4	5	5	5	6	6	7	7	7	7
	11		2	3	4	4	5	5	6	6	7	7	7	8	8
	12	2	2	3	4	4	5	6	6	7	7	7	8	8	8
	13	2	2	3	4	5	5	6	6	7	7	8	8	9	9
	14	2	2	3	4	5	5	6	7	7	8	8	9	9	9
	15	2	3	3	4	5	6	6	7	7	8	8	9	9	10

Critical Upper Reject Values of r for Runs Test ($1-\alpha/2=0.975$)

		n2											
		4	5	6	7	8	9	10	11	12	13	14	15
	4		9	9									
	5	9	10	10	11	11							
	6	9	10	11	12	12	13	13	13	13			
	7		11	12	13	13	14	14	14	14	15	15	15
	8		11	12	13	14	14	15	15	16	16	16	16
n1	9			13	14	14	15	16	16	16	17	17	18
	10			13	14	15	16	16	17	17	18	18	18
	11			13	14	15	16	17	17	18	19	19	19
	12			13	14	16	16	17	18	19	19	20	20
	13				15	16	17	18	19	19	20	20	21
	14				15	16	17	18	19	20	20	21	22
	15				15	16	18	18	19	20	21	22	22

Figure 2: Critical Reject Values for the Runs Test